Two Proportions: Difference, RR and OR

Consider two proportions p_1 and p_2: e.g., p_1 is Risk of Disease when Not Exposed to a Risk Factor, p_2 is Risk of Disease when Exposed to the Risk Factor. Between p_2 and p_1 there are the relationships Difference Δ; Risk Ratio (RR) λ; Odds Ratio (OR) ρ:

$\Delta = p_2 - p_1$	$p_2 = p_1 + \Delta$	*Difference*
$\lambda = \dfrac{p_2}{p_1}$	$p_2 = \lambda p_1$	*RR*
$\rho = \dfrac{p_2/(1-p_2)}{p_1/(1-p_1)}$	$p_2 = \dfrac{\rho p_1}{1+(\rho-1)p_1}$	*OR*

This note illustrates the interactions between Difference, RR, and OR. The diagram was drawn for $p_1 = 0.45$ and $p_2 = 0.75$ (at the box), so that $\Delta = 0.3$ (Difference), $\lambda = 5/3$ (RR), and $\rho = 11/3$ (OR). It will be assumed below that $\Delta > 0$ (else reverse p_1 and p_2), so that $\lambda > 1$ and $\rho > 1$. The equations for Δ and λ can be solved for p_1 and p_2 to give

$$p_1 = \Delta/(\lambda - 1) \qquad p_2 = \lambda\Delta/(\lambda - 1)$$

Derivations of other results stated below are given in the **Appendix**.

For ***Constant Difference***, (p_1, p_2) may lie anywhere on the *Difference line* from $(0, \Delta)$ to $(1-\Delta, 1)$, so $0 \le p_1 \le 1-\Delta$ and $\Delta \le p_2 \le 1$.

For ***Constant RR***, (p_1, p_2) may lie anywhere on the *RR line* from $(0, 0)$ to $(1/\lambda, 1)$. so $0 \le p_1 \le 1/\lambda$ and $0 \le p_2 \le 1$. Since this line has slope $\lambda > 1$, it intersects the *Difference line* $p_2 = p_1 + \Delta$ at (p_1, p_2).

For ***Constant OR***, the relationship between p_1 and p_1 is non-linear; (p_1, p_2) may lie anywhere on the *OR curve*: $0 < \{p_1, p_2\} < 1$.

The *RR line* $p_2 = \lambda p_1$ meets the *OR curve* at the unique point (p_1, p_2).

The *Difference line* $p_2 = p_1 + \Delta$ meets the *OR curve* at two points, at values of p_1 satisfying a quadratic equation, one of which is (p_1, p_2).

The gradient of the *OR curve* is $\rho/(1+(\rho-1)p_1)^2$, and at $p_1 = 0$ this has value ρ (shown in the diagram as the dashed line $p_2 = \rho p_1$). Hence the *RR line* cannot be steeper than this, so for given $\mathrm{OR} = \rho$ we must have $\mathrm{RR} = \lambda < \rho$. (***Note*** At $p_1 = 1$, the gradient is $1/\rho$).

For given $\mathrm{OR} = \rho$, the *Difference line* $p_2 = p_1 + \Delta$ can be moved upwards (increasing Δ) until it is just tangential to the *OR curve*, meeting it at a single point (marked ▪ on the diagram). This occurs

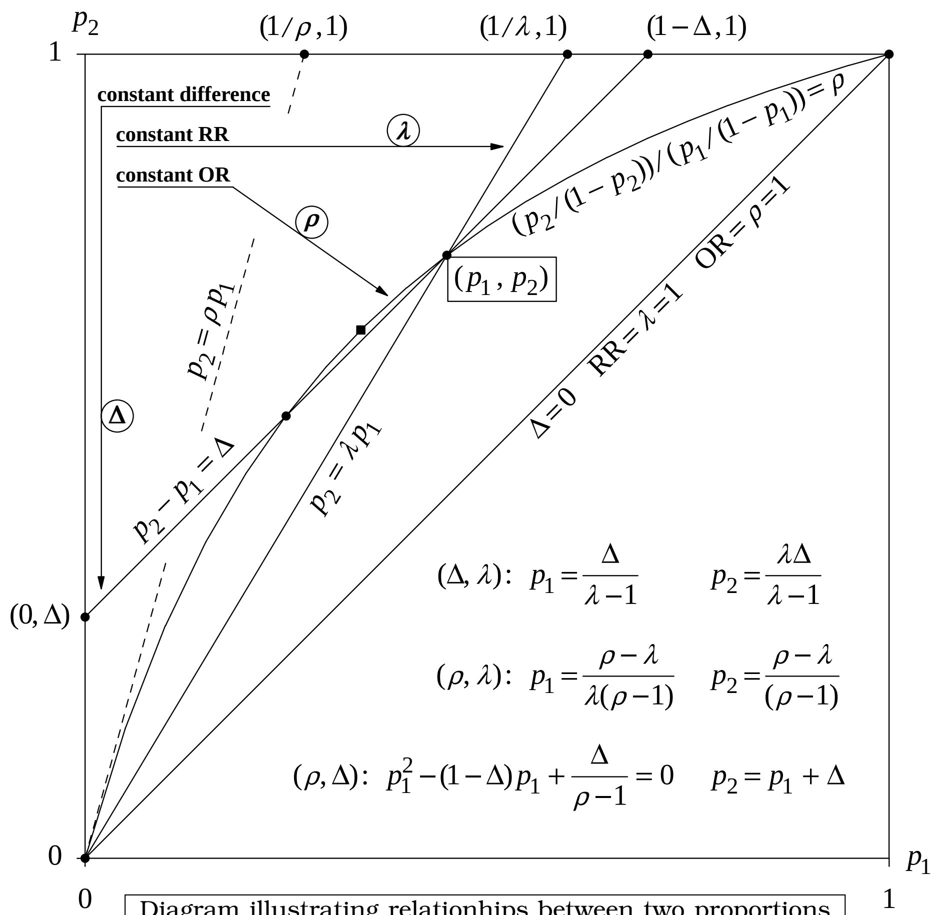

Diagram illustrating relationhips between two proportions p_1, p_2, their Difference $\Delta = p_1 - p_2$, their Relative Risk $\lambda = p_1/p_2$, and their Odds Ratio $\rho = (p_2/(1-p_2))/(p_1/(1-p_1))$.

when $(\rho-1)(\Delta-1)^2 = 4\Delta$, and the tangent is at $p_1 = 1/(\sqrt{\rho}+1)$ so then $p_2 = \sqrt{\rho}/(\sqrt{\rho}+1)$, $\lambda = \sqrt{\rho}$, and $\Delta = (\sqrt{\rho}-1)/(\sqrt{\rho}+1)$. The gradient here is 1. This is the largest possible value of Δ for a given value ρ of the OR.

The proportions p_1, p_2 are uniquely determined by their Difference Δ and RR λ, or by their RR λ and OR ρ. However, when their Difference Δ and OR ρ are given, there are in general two possibilities.

Appendix

It is assumed throughout that $\lambda > 1$ and $\rho > 1$ (i.e. $p_2 > p_1$, so $\Delta > 0$).

$$\Delta = p_2 - p_1 \qquad p_2 = p_1 + \Delta \qquad \textit{Difference} \qquad \text{A[1]}$$

$$\lambda = \frac{p_2}{p_1} \qquad p_2 = \lambda p_1 \qquad \textit{RR} \qquad \text{A[2]}$$

$$\rho = \frac{p_2/(1-p_2)}{p_1/(1-p_1)} \qquad p_2 = \frac{\rho p_1}{1+(\rho-1)p_1} \qquad \textit{OR} \qquad \text{A[3]}$$

Proof of A[3]

From the definition of ρ on the left of A[3], the expression for p_2 on the right follows easily, since

$$\rho p_1(1-p_2) = (1-p_1)p_2 \quad \text{so} \quad (1 - p_1 + \rho p_1)p_2 = \rho p_1$$

Given Difference and RR

From A[1] and A[2], $\lambda p_1 = p_1 + \Delta$, so for given difference Δ and RR λ

$$p_1 = \frac{\Delta}{\lambda - 1} \qquad p_2 = \frac{\lambda\Delta}{\lambda-1} \qquad \text{A[4]}$$

From A[4], since $0 < \{p_1, p_2\} < 1$, we must have $(\lambda-1) > \Delta$ and $\lambda(1-\Delta) > 1$. For $\Delta > 0$, the second of these is the stronger, so the only valid specifications of (Δ, λ) are such that $\lambda > 1/(1-\Delta)$ (see diagram). For any such specification, A[4] yields a unique valid pair (p_1, p_2).

Given RR and OR

From A[2] and A[3], for given RR λ and OR ρ

$$\frac{\rho p_1}{1+(\rho-1)p_1} = \lambda p_1 \quad \text{so} \quad \rho = \lambda(1+(\rho-1)p_1)$$

$$p_1 = \frac{\rho-\lambda}{\lambda(\rho-1)} \qquad p_2 = \frac{\rho-\lambda}{\rho-1} \qquad \text{A[5]}$$

From A[5], we must have $\lambda \le \rho$, otherwise $p_1 < 0$ and $p_2 < 0$. For $\lambda > 1$ this follows anyway, since

$$\lambda > 1 \iff \frac{p_2}{p_1} > 1 \iff \frac{1-p_2}{1-p_1} < 1 \iff \rho = \frac{p_2/p_1}{(1-p_2)/(1-p_1)} > \frac{p_2}{p_1} = \lambda$$

For $\lambda > 1$ it also follows that $\rho - \lambda < \rho - 1$ so, in A[5], $p_1 < 1$ and $p_2 < 1$. Hence, given any values of ρ and λ such that $1 < \lambda < \rho$, A[5] yields a unique valid pair (p_1, p_2).

Given Difference and OR

Given Δ, $p_2 = p_1 + \Delta$, so it follows from A[3] that

$$\frac{\rho p_1}{1+(\rho-1)p_1} = p_1 + \Delta \quad \text{so} \quad \rho p_1 = (1+(\rho-1)p_1)(p_1+\Delta)$$

$$\rho p_1 = p_1 + \Delta + (\rho-1)p_1^2 + (\rho-1)\Delta p_1$$

$$(\rho-1)p_1^2 + (1-\rho+(\rho-1)\Delta)p_1 + \Delta = 0$$

$$p_1^2 - (1-\Delta)p_1 + \frac{\Delta}{\rho-1} = 0 \qquad \text{A[6]}$$

This quadratic equation $ap_1^2 + bp_1 + c = 0$, with $a = 1$, $b = -(1-\Delta)$, $c = \Delta/(\rho-1)$, has in general **two solutions** $p_1 = \left(-b \pm \sqrt{b^2-4ac}\right)/(2a)$:

$$p_1 = \frac{1-\Delta}{2} \pm \frac{1}{2}\left((1-\Delta)^2 - \frac{4\Delta}{\rho-1}\right)^{\frac{1}{2}} \qquad p_2 = p_1 + \Delta \qquad \text{A[7]}$$

This has **coincident solutions** when $b^2 = 4ac$, and then $p_1 = -\frac{1}{2}b$, so

$$(\rho-1)(1-\Delta)^2 = 4\Delta \qquad (\rho-1)\Delta^2 - (2(\rho-1)+4)\Delta + (\rho-1) = 0$$

$$\Delta^2 - 2\frac{\rho+1}{\rho-1} + 1 = 0 \qquad \left(\Delta - \frac{\sqrt{\rho}+1}{\sqrt{\rho}-1}\right)\left(\Delta - \frac{\sqrt{\rho}-1}{\sqrt{\rho}+1}\right) = 0$$

The solution $\Delta = (\sqrt{\rho}+1)/(\sqrt{\rho}-1)$ is excluded because it gives $\Delta > 1$, since $\Delta \le 1$. Hence, since in this case $p_1 = \frac{1}{2}(1-\Delta)$, using $p_2 = p_1 + \Delta$,

$$\Delta = \frac{\sqrt{\rho}-1}{\sqrt{\rho}+1} \qquad p_1 = \frac{1}{2}\left(1 - \frac{\sqrt{\rho}-1}{\sqrt{\rho}+1}\right) = \frac{1}{\sqrt{\rho}+1} \qquad p_2 = \frac{\sqrt{\rho}}{\sqrt{\rho}+1} \qquad \text{A[8]}$$

for **coincident solutions**, and then $RR = p_2/p_1 = \lambda = \sqrt{\rho}$. The value of Δ in A[8] is the maximum value of Δ (see diagram) that is compatible with a given OR ρ; and it can be solved for ρ to give the minimum value $\rho_{\min}(\Delta)$ of ρ that is compatible with a given value of Δ:

$$\Delta\sqrt{\rho} + \Delta = \sqrt{\rho} - 1 \qquad (1-\Delta)\sqrt{\rho} = 1 + \Delta \qquad \rho_{\min}(\Delta) = \left(\frac{1+\Delta}{1-\Delta}\right)^2 \qquad \text{A[9]}$$

Gradient of p_2 *for Given OR*

From A[3], $\quad p_2 = \dfrac{\rho p_1}{1+(\rho-1)p_1}$

$$\text{so} \quad \frac{dp_2}{dp_1} = \frac{\rho}{1+(\rho-1)p_1} - \frac{\rho(\rho-1)p_1}{(1+(\rho-1)p_1)^2} = \frac{\rho}{(1+(\rho-1)p_1)^2} \qquad \text{A[10]}$$

in general, for all values of p_1 and p_2 with the same OR ρ.